J. DIFFERENTIAL GEOMETRY
6 (1972) 421-436

CONVEXITY CONDITIONS RELATED WITH 1/2 ESTIMATE IN BOUNDARY PROBLEMS WITH SIMPLE CHARACTERISTICS. I

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The classical works of C. B. Morrey and J. J. Kohn say that 1/2 estimate in $\bar{\partial}$-Neumann problem holds on strongly pseudo-convex manifolds. The main aim of the present paper is to find similar convexity conditions which imply 1/2 estimate in more general boundary value problems for elliptic differential operators (under some simplicity assumptions on characteristics of boundary problems). The convexity conditions will be expressed as algebraic inequalities (in a general sense) on the boundary involving 1-jets of symbols of the elliptic operators. To simplify the presentation, we limit ourselves to the case of operators of order 1.

In Chapter I, following Hörmander we reduce the problem to another one on the boundary; this part, except for a few slight changes in presentation (an influence of [10]) and some corollaries, is entirely copied from [3]. At the end of the chapter, we state a theorem (cf. [2]) which is based on K. Friedrich's proof of P. D. Lax–L. Nirenberg theorems on generalized Garding inequality; the author is indebted to Professor H. Kumanogo who communicated this to him. The theorem is the main tool used in Chapter II where we derive sufficient conditions (Proposition 2.3) for 1/2 estimate on pseudo-differential operators on manifolds without boundary under the assumptions that characteristics are smooth, nondegenerate, and of zero fiber dimension (Definitions 2.2, 2.3, and 2.4). These conditions are similar to those obtained by Hörmander (the latter are also necessary) in [3]. But from ours it is a little easier to obtain algebraic sufficient conditions which are studied in Chapter III.

1. Reduction to the boundary

Let $Y^{\#}$ be a manifold, and denote by Y an open submanifold of $Y^{\#}$. Assume that Y has a smooth boundary manifold $M \subset Y^{\#}$ such that $Y \cup M$ is compact, and further assume that for any x in M there is a neighborhood $U^{\#}$ of x in $Y^{\#}$ such that $U^{\#} \cap M$ divides $U^{\#}$ into two connected parts, one inside Y and another

Received August 11, 1971. Partially supported by the National Science Foundation under Grant No. GP–8988. Due to its length this paper is being published in two parts; part II will appear in the first issue of the next volume.

outside. Then there are a compact neighborhood N of M in $Y^{\#}$ and a diffeomorphism

$$(1) \qquad h: N \to M \times I ,$$

where I is the closed interval $[-\varepsilon, \varepsilon]$ with a positive ε, such that $N \cap Y$ is mapped onto $M \times [-\varepsilon, 0]$. We fix once and for all such h throughout this paper. In order to construct such a diffeomorphism h, we may introduce a Riemannian metric on $Y^{\#}$ and draw geodesics from points in M.

Composing the projection to I and h, we have a function $N \to \boldsymbol{R}$ of class C^{∞}, which we denote by x_n (where n is the dimension of Y). x_n is called the distance function to M. The cotangent vector dx_n at x in N is called the outward unit normal cotangent vector at x, and the cotangent vector field dx_n on N is called the outward unit normal cotangent vector field on N. If we use a Riemannian metric on $Y^{\#}$ to construct h and use a suitable parametrization of geodesics, our outward unit normal cotangent vector at a point x in Y is the outward unit cotangent vector at x normal to the boundary of Y in the sense of Riemannian geometry. Let t be the standard chart of I. Then the vector field $\partial/\partial t$ can be considered as a vector field on $M \times I$, and therefore can be transplanted on N by h. We call this vector field the unit normal tangent vector field on N and will be denoted by $\partial/\partial \boldsymbol{n}$. Let $(x_1, \cdots, x_{n-1})$ be a chart of M with domain U. Then, by using the product structure of N, we can regard $x_1, \cdots, x_{n-1}$ as functions on $U^{\#} = h^{-1}(U \times I^i)$, where $I^i = (-\varepsilon, \varepsilon)$. Moreover, $(x_1, \cdots, x_{n-1}, x_n)$ is a chart of $Y^{\#}$ with domain $U^{\#}$, and is called a distinguished chart. It this is the case, the restriction of $\partial/\partial \boldsymbol{n}$ to $U^{\#}$ is $\partial/\partial x_n$.

For a vector bundle E over $Y^{\#}$, $C_0^{\infty}(Y^{\#}, E)$ denotes the vector space of smooth sections of E over $Y^{\#}$ with compact support and also $C^{\infty}(Y^{\#}, E)$ with no restriction on support. For $u \in C_0^{\infty}(Y^{\#}, E)$, we denote by $\|u\|_s$ ($s \in \boldsymbol{R}$) a Sobolev s-norm of u, which is defined by picking a locally finite covering $\{W_\alpha\}$ by open sets (where $\overline{W}_\alpha$ are compact) together with (1) a partition of unity $\{f_\alpha^2\}$ with respect to the covering, (2) a chart of $Y^{\#}$ with domain W_α, and (3) a local trivialization of E over W_α. Then

$$\|u\|_s^2 = \sum_\alpha \|f_\alpha u\|_s^2 ,$$

where $f_\alpha u$ is now considered as a system of C^{∞} functions on $\boldsymbol{R}^n$ by means of the chart and local trivializations, and $\|f_\alpha u\|_s^2$ denotes the usual Sobolev s-norm. We fix once and for all a Sobolev s-norm on $Y^{\#}$ throughout this paper. We denote by $\mathscr{H}_0^s(Y^{\#}, E)$ the completion of $C_0^{\infty}(Y^{\#}, E)$ with respect to $\| \ \|_s$, and by $C^{\infty}(Y^{cl}, E)$ the vector space of smooth sections of E over $Y^{cl} = Y \cup M$, the closure of Y in $Y^{\#}$. For $u \in C^{\infty}(Y^{cl}, E)$ we define $\|u\|_s$ as the infimum of $\|v\|_s$ where v runs through all C^{∞} extensions of u with support in $Y \cup N$. The equivalence class of the norm $\| \ \|_s$ on $C^{\infty}(Y^{cl}, E)$ is independent of the choice of Sobolev norm of $C_0^{\infty}(Y^{\#}, E)$ and also of the choice of N as above. We also denote by $\mathscr{H}^s(Y^{cl}, E)$ the completion of $C^\infty(Y^{cl}, E)$ with respect to $\| \ \|_s$.

Let χ be the characteristic function of Y, i.e., $\chi(x) = 1$ for x in Y and $\chi(x) = 0$ otherwise. χ is in $\mathscr{H}_0^0(Y^\#, C)$ where C denotes the trivial vector bundle of fiber dimension 1. Let g be a C^∞ function with support in N such that $g = 1$ on a neighborhood of M in $Y^\#$. For any $u \in C^\infty(M, C)$ denote by $u^\#$ a C^∞ extension of u to $Y^\#$ with compact support such that $u^\#(h^{-1}(x, t)) = u(x)$ for all x in M and t in I. Then $(g\partial/\partial\boldsymbol{n})^r(\chi u^\#)$ is in $\mathscr{H}_0^{-r}(Y^\#)$, since a differential operator of order 1 induces a continuous mapping $\mathscr{H}_0^s(Y^\#, C)$ into $\mathscr{H}_0^{s-1}(Y^\#, C)$. It is clear that it is independent of the choice of g and $u^\#$ as above. We set

$$u \otimes \delta^r = -(g\partial/\partial\boldsymbol{n})^r(\chi u^\#) \in \mathscr{H}_0^{-r}(Y^\#, C)$$

for $u \in C^\infty(M, C)$.

We are going to define $u \otimes \delta^r$ for u in $C^\infty(M, E)$. For this purpose we make a number of conventions. Let P be a differential operator $C^\infty(Y^\#, E) \to C^\infty(Y^\#, F)$, where F is a vector bundle over $Y^\#$. For a cotangent vector ξ to $Y^\#$ at a point x we denote by $\sigma(P, \xi)$ the leading symbol of P at ξ. Thus $\sigma(P, \xi)$ is a linear mapping of E_x into F_x where a suffix x indicates the fiber over x. Now it is easy to construct a differential operator $(\partial/\partial\boldsymbol{n})_E \colon C^\infty(N, E) \to C^\infty(N, E)$ such that

$$\sigma((\partial/\partial\boldsymbol{n})_E, \xi) = \sigma(\partial/\partial\boldsymbol{n}, \xi)I_x ,$$

where I_x denotes the identity mapping of E_x. There are many $(\partial/\partial\boldsymbol{n})_E$ with the above property, but they differ only by operators of order 0. Whenever we have a vector bundle E over N, we fix $(\partial/\partial\boldsymbol{n})_E$ once and for all and call it the lifting of $\partial/\partial\boldsymbol{n}$ to E. Let $(x_1, \cdots, x_n)$ be a distinguished chart on a domain $U^\#$, and pick a local C^∞ trivialization of E over $U^\#$, say $E\,|\,U^\# \approx U^\# \times E_0$. Then a section v of E over $U^\#$ can be represented by a function, say $v(x)$, with values in a vector space E_0, and we may write $(\partial/\partial\boldsymbol{n})_E v(x) = (\partial/\partial x_n + a(x))v(x)$, where $a(x)$ is a function with values in the space of endomorphism of E_0. If we change the local trivialization by applying $g(x)^{-1}$, where $g(x)$ is in $GL(E_0) =$ the group of automorphisms of E_0, then the expression of $(\partial/\partial\boldsymbol{n})_E$ changes to

$$g(x)^{-1}(\partial/\partial x_n + a(x))g(x) = \partial/\partial x_n + (g(x)^{-1}\partial g/\partial x_n + g(x)^{-1}a(x)g(x)) .$$

On the other hand we can always solve the equation $\partial g(x)/\partial x_n = -a(x)g(x)$ (shrinking N and U if necessary). This means that we can always find a local trivialization of E over $U^\#$ such that the expression of $(\partial/\partial\boldsymbol{n})_E$ is equal to $\partial/\partial x_n$. Thus, under these choices of local trivialization which are said to be distinguished, $(\partial/\partial\boldsymbol{n})_E$ can be considered as normal derivation. For $u \in C^\infty(M, E)$, let $u^\# \in C_0^\infty(Y^\#, E)$ be such that $(\partial/\partial\boldsymbol{n})_E u^\# = 0$ on N. $u^\#\,|\,N$ is uniquely determined, and $u^\#(x', 0) = u^\#(x', x_n)$ provided we express $u^\#$ in terms of a distinguished chart (x', x_n), $x' = (x_1, \cdots, x_{n-1})$, and a distinguished local trivialization of E. Then as before we set

$$u \otimes \delta^r = -(g\partial/\partial n)_E^r(\chi u^\#) \in \mathscr{H}_0^{-r}(Y^\#, E) .$$

Let $Q: C_0^\infty(Y^\#, E) \to C^\infty(Y^\#, F)$ be a pseudo-differential operator of order $-m$. If f is in $C_0^\infty(Y^\#, C)$, fQ can be extended to a bounded operator: $\mathscr{H}_0^s(Y^\#, E) \to \mathscr{H}_0^{s+m}(Y^\#, F)$. In particular $fQ(u \otimes \delta^r)$ is well defined and is in $\mathscr{H}_0^{-r+m}(Y^\#, F)$. By the quasi-local property of pseudo-differential operators, $fQ(u \otimes \delta^r)$ is of class C^∞ outside of M. Thus $Q(u \otimes \delta^r)$ is a C^∞ function on the complement of M in $Y^\#$. We say that $Q_-^r(u)$ (resp. $Q_+^r(u)$) exists when $Q(u \otimes \delta^r) | Y$ (resp. $Q(u \otimes \delta^r) | (Y \cup M)^c$ where c indicates complement in $Y^\#$) can be extended to a C^∞ section of E over $Y^\#$. If this is the case, its restriction to M will be denoted by $Q_-^r(u)$ (resp. $Q_-^r(u)$). Thus, for $u \in C^\infty(M, E)$, $Q_-^r(u)$ and $Q_+^r(u)$ are $C^\infty(M, F)$ when they exist.

Let $q(x, \xi)$ be the matrix of symbols of Q with respect to a chart (x) of $Y^\#$ and a local trivialization of E over the domain of the chart. In the following, when we say that $q(x, \xi)$ is the matrix of symbols of Q, we always assume that we pick a chart (x) and a local trivialization without mentioning them. We always assume that $q(x, \xi)$, for ξ outside of a ball of a sufficiently large radius, can be extended to a matrix of meromorphic functions in a complex vector $\zeta \in (T_x^*Y^\#) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ for each x. We refer the following theorem to [3, p. 141].

Theorem 1.1. *Under a suitable assumption on Q which is satisfied when Q is a parametrix of an elliptic differential operator, we have the following*:

1) *$Q_+^r(u)$ and $Q_-^r(u)$ exist for all u in $C^\infty(M, E)$. Thus Q_+^r and Q_-^r can be considered as a map of $C^\infty(M, E)$ into $C^\infty(M, F)$.*

2) *Q_+^r and Q_-^r are pseudo-differential operators of order $r - m$ where $-m$ is the order of Q.*

3) *Assume that (x) is a distinguished chart of $Y^\#$ and that we use a distinguished local trivialization of E. If we use (x', ξ'), where $x' = (x_1, \cdots, x_{n-1}, 0)$ and $\xi' = (\xi_1, \cdots, \xi_{n-1}, 0)$, as a chart of T^*M, then the matrix of symbols $q^+(x', \xi')$ of Q_+^1 is given by*

$$q^+(x', \xi') = \frac{1}{2\pi} \int_{\Gamma_+(R_{\xi'})} q(x', \xi' + ze_n)dz ,$$

where $e_n = (0, \cdots, 0, 1)$, $\Gamma_+(R)$ is the contour in the complex plane consisting of the upper half circle of radius R with center zero together with the interval $[-R, R]$ oriented anti-clockwise, and where $R_{\xi'}$ is a sufficiently large number so that all poles of $q(x', \xi' + ze_n)$ in the upper half plane are inside of $\Gamma_+(R_{\xi'})$. Similarly, the matrix of symbols $q^-(x', \xi')$ of Q_-^1 is obtained by replacing $+$ (resp. upper, anti-clockwise) by $-$ (resp. lower, clockwise) in the above. Namely, we introduce orientations on $\Gamma_+(R)$ and $\Gamma_-(R)$ which induce the orientation "from $-$ to $+$" on $[-R, R]$.

Remark. We have similar expressions for the symbol of Q_+^r and Q_-^r for all $r = 1, 2, \cdots$. However, we omitted them because we do not use them in this paper.

Let $P\colon C^\infty(Y^\sharp, E) \to C^\infty(Y^\sharp, F)$ be a differential operator of order 1. For $\xi \in T_x^* Y^\sharp$, we denote the leading symbol mapping of P at ξ by $\sigma(P, \xi)$, which is a linear mapping of E_x into F_x and linear in $\xi \in T_x^* Y^\sharp$. Thus we have a vector bundle mapping (which may not be of constant rank)

$$\sigma_b(P)\colon E|M \to F|M$$

defined by the formula: $\sigma_b(P)|E_x = \sigma(P, dx_n)$. Let $(x) = (x_1, \cdots, x_n)$ be a distinguished chart. In terms of a local trivialization of E over the domain $U^\sharp$, we can consider $\partial/\partial x_j$ as a differential operator acting on $C^\infty(U^\sharp, E)$ and may write $P = \sum_{j=1}^n (1/i)\sigma(P, dx_j)(\partial/\partial x_j) + \sigma_0(x)$. On the other hand, dx_n (resp. $\partial/\partial x_n$) is independent of the choice of a distinguished chart, i.e., it is the unit normal tangent (resp. cotangent) vector field. Hence

$$P - \frac{1}{i}\sigma_b(P)(\partial/\partial n) = \frac{1}{i}\sum_{n=1}^{n-1} \sigma(P, dx_r)\partial/\partial x_r + \sigma_0(x)$$

is a well defined differential operator on N, and induces a differential operator on M, which will be denoted by

$$P_t\colon C^\infty(M, E) \to C^\infty(M, F) ,$$

and is called the tangential part on M of P. By definition

$$(3) \qquad P\chi u - \chi P u = i(\sigma_b(P)u) \otimes \delta^1$$

($u \in C_0^\infty(Y^\sharp, E)$). Assume further that P is elliptic and $\dim E = \dim F$. Then there is a pseudo-differential operator $Q\colon C_0^\infty(Y^\sharp, F) \to C^\infty(Y^\sharp, E)$ of order -1 such that

$$(4) \qquad Q \circ P = I_E + K_E , \qquad P \circ Q = I_F + K_F ,$$

where I_E (resp. I_F) is the identity map of $C_0^\infty(Y^\sharp, E)$ (resp. of $C_0^\infty(Y^\sharp, F)$), and K_E (resp. K_F) is a pseudo-differential operator of order $-\infty$ mapping $C_0^\infty(Y^\sharp, E)$ into $C^\infty(Y^\sharp, E)$ (resp. $C_0^\infty(Y^\sharp, F)$ into $C^\infty(Y^\sharp, F)$). Set

$$(5) \qquad A_+ = -iQ_+^1 \circ \sigma_b(P) , \qquad A_- = iQ_-^1 \circ \sigma_b(P) .$$

A_+ and A_- are pseudo-differential operators of order 0 which map $C^\infty(M, E)$ into itself. Let $K_b\colon C^\infty(M, E) \to C^\infty(Y^\sharp, E)|Y$ be defined by

$$(6) \qquad K_b(\gamma) = K_F(\sigma_b(P)\gamma \otimes \delta^1)|Y .$$

K_b is an operator of order $-\infty$. The significance of A_+ and A_- is given in

Theorem 1.2. *Let $P\colon C^\infty(Y^\sharp, E) \to C^\infty(Y^\sharp, F)$ be an elliptic differential operator of order* 1, *assume that* $\dim E = \dim F$, *and let A_+, A_- be defined as above. Then*:

1) $A_- + A_+ =$ *the identity map*;

2) *if* $u \in C^\infty(Y^\#, E)$, $Pu = 0$ *on* Y, *and* $K_E\chi u = 0$ *on* Y, *then* $A_+(bu) = 0$ *and* $K_b(bu) = 0$, *where* χ *is the characteristic function of* Y *and* bu *denotes the restriction of* u *to* M;

3) *if* $\gamma \in C^\infty(M, E)$, $A_+(\gamma) = 0$, *and* $K_b(\gamma) = 0$, *then there is* $u \in C^\infty(Y^\#, E)$ *such that* $bu = \gamma$, $Pu = 0$ *on* Y, *and that* $K_E\chi u = 0$. *Moreover,* u *as above is uniquely determined on* Y *and is given by* $u|Y = iQ(P_t\gamma \otimes \delta^1)|Y$.

Proof. 1) Applying Q to the formula (3), we find that

$$\chi u + K_E\chi u - Q\chi Pu = iQ(\sigma_b(P)u \otimes \delta^1) . \tag{7}$$

Since K_E is of order $-\infty$, $K_E\chi u$ is smooth on $Y^\#$. We can prove without much difficulty that $Q\chi Pu$ is continuous on $h^{-1}(x \times I)$ for each x in M, where h is given in the formula (1). Hence

$$A_-(bu) + A_+(bu) = \lim_{x_n \uparrow 0} \chi u - \lim_{x_n \downarrow 0} \chi u = bu .$$

2) By our assumption, (7) implies that

$$u|Y = iQ(\sigma_b(P)u \otimes \delta^1)|Y . \tag{8}$$

Therefore, by (5), $A_-(bu) = bu$, so that, by 1), $A_+(bu) = 0$. Applying P to (8), we find that

$$\begin{aligned} Pu|Y = iPQ(\sigma_b(P)u \otimes \delta^1)|Y &= i(\sigma_b(P)u \otimes \delta^1)|Y \\ &+ iK_F(\sigma_b(P)u \otimes \delta^1)|Y = iK_b(\gamma) . \end{aligned}$$

Since $Pu|Y = 0$, we see that $K_b(\gamma) = 0$.

3) Take $u \in C_0^\infty(Y^\#, E)$ such that $u|Y = iQ(\sigma_b(P)\gamma \otimes \delta^1)|Y$. Such u exists because Q_-^1 exists. Then by definition, $bu = A_-\gamma$. Since $A_+\gamma = 0$ by assumption, it follows by 1) that $\gamma = A_-\gamma + A_+\gamma = bu$. Applying P to the above defining formula of $u|Y$, we see that $Pu|Y = iPQ(\sigma_b(P)\gamma \otimes \delta^1)|Y = iK_F(\sigma_b(P)\gamma \otimes \delta^1)|Y = iK_b(\gamma) = 0$. By (7) it then follows that $\chi u + K_E\chi u = iQ(\sigma_b(P)u \otimes \delta^1) = iQ(\sigma_b(P)\gamma \otimes \delta^1)$. Hence by the definition of u, $K_E\chi u = 0$ on Y. This finishes the proof of 3) except the uniqueness of $u|Y$. To see this, assume that $Pu|Y = 0, K_E\chi u = 0$, and that $bu = 0$. Then by (7), $\chi u = 0$. Thus the uniqueness is also proved.

Remark. We can construct operators similar to A_+ and A_- for a determined elliptic operator P of order $m > 1$. However, we have to take into account normal derivatives up to order $m - 1$ at M, and the construction is a little more involved ([1], [3], [10]). Also, we can generalize the result to the case where P is a pseudo-differential operator provided P is a differential operator on N and there is a neighborhood N' of M such that Supp $Pu \cap N' = \emptyset$ for u with support outside N.

Our theorem says that, up to errors of order $-\infty$, the condition $Pu = 0$ on Y is more or less equivalent to the condition $A_+bu = 0$ on M. Hence we may feel that properties of P on Y are fairly faithfully reflected by properties of A_+. The following is one such example (cf. [3]):

Theorem 1.3. *Let $B \subseteq C^\infty(M, E)$ be a vector subspace, s be a real number, and $0 < a < 1$. Then the following two conditions are equivalent:*

1) *There is a constant $C > 0$ such that*

$$\|Pu\|_s + \|u\|_s \geq C\|u\|_{s+a}$$

for all $u \in C^\infty(\overline{Y}, E)$ with boundary condition $bu \in B$.

2) *There is a constant $C' > 0$ such that*

$$\|A_+\gamma\|_{s+\frac{1}{2}} + \|\gamma\|_{s-\frac{1}{2}} \geq C'\|\gamma\|_{s+a-\frac{1}{2}}$$

for all $\gamma \in B$.

$\sigma(P, \xi)$ is in the vector space Hom (E_x, F_x) *of linear mappings over* $\boldsymbol{C}$ *of E_x into F_x. Then $\sigma(Q, \xi) = \sigma(P, \xi)^{-1} \in$* Hom (F_x, E_x) *where $\sigma(Q, \xi)$ denotes the leading symbol of Q at ξ. For $\zeta \in T_xY^\# \otimes_{\boldsymbol{R}} \boldsymbol{C}$, denote by $\sigma(P, \zeta)$ the extension of the linear mapping $\xi \to \sigma(P, \xi) \in$* Hom (E_x, F_x) *to a* Hom (E_x, F_x)*-valued analytic function in ξ. Identify $T_x^*M (x \in M)$ as the subspace of $T_x^*Y^\#$ orthogonal to $\partial/\partial \boldsymbol{n}$ at x. Thus*

$$T_x^*M \subseteq T_x^*Y^\# . \tag{9}$$

Then for $\xi' \in T_x^*M$ the leading symbol $\sigma(A_+, \xi')$ is given, because of its construction and Theorem 1.1, *by*

$$\sigma(A_+, \xi') = \frac{1}{2\pi i}\int_{\Gamma_+(R_{\xi'})} \sigma(P, \xi' + z\boldsymbol{n}(x))^{-1}\sigma(P, \boldsymbol{n}(x))dz \tag{10}$$

where $\boldsymbol{n}(x)$ is the outward unit normal cotangent vector at x, and the integral is taken as a Hom (E_x, E_x)*-valued meromorphic function in z. Thus to calculate the leading symbol of A_+ we may use any chart, not necessarily a distinguished one.*

Example. Assume that, with respect to a choice of hermitian metric on E and F, $\sigma(P, \xi)^*\sigma(P, \xi) = |\xi|^2$ (identity map) where $|\xi|^2$ is a metric on T^*M. Assume further that $|\boldsymbol{n}(x)| = 1$ and $\langle \xi', \boldsymbol{n}(x) \rangle = 0$ for all $\xi' \in T_x^*M$. Then, we may write $\xi = \xi' + \xi_n \boldsymbol{n}(x)$ and $|\xi|^2 = |\xi'|^2 + \xi_n^2$. Therefore

$$\sigma(P, \xi)^{-1} = (|\xi'|^2 + \xi_n^2)^{-1}\sigma(P, \xi)^* ,$$

$$\sigma(P, \xi' + z\boldsymbol{n}(x))^{-1}\sigma(P, \boldsymbol{n}(x)) = \frac{1}{z^2 + |\xi'|^2}(\sigma(P, \xi')^*\sigma(P, \boldsymbol{n}(x)) + zI) ,$$

where I denotes the identity map. Therefore, by (10),

$$\sigma(A_+, \xi') = \tfrac{1}{2}(I - i|\xi'|^{-1}\sigma(P, \xi')^*\sigma(P, \boldsymbol{n}(x))) , \qquad (\xi' \in T_x^*M) . \tag{11}$$

We are going to describe another way of determining $\sigma(A_+, \xi')$ algebraically. We note first that the linear transformation $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi')$, where $\xi' \in T_x^*M$, does not have any real eigenvalue. Namely, if $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi')u = \gamma u$ with a real number γ, then $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi' - \gamma\boldsymbol{n}(x))u = 0$ and hence $u = 0$ by the ellipticity of P. Let

$$\mu_1(\xi'), \cdots, \mu_k(\xi') \qquad (\text{resp. } \lambda_1(\xi'), \cdots, \lambda_h(\xi'))$$

be distinct eigenvalues of $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi')$ with positive imaginary part (resp. with negative imaginary part). Denote by $E_+^i(\xi')$ (resp. $E_-^i(\xi')$) be the kernel of $(\sigma(P, n(x))^{-1}\sigma(P, \xi') - \lambda_i(\xi')I)^l$ (resp. of $(\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi') - \mu_j(\xi')I)^l$) for sufficiently large l. Set

$$E_+(\xi') = E_+^1(\xi') + \cdots + E_+^k(\xi') , \qquad E_-(\xi') = E_-^1(\xi') + \cdots + E_-^h(\xi') .$$

Let $b_+(\xi')$ (resp. $b_-(\xi')$) be the projection to $E_+(\xi')$ (resp. $E_-(\xi')$) with respect to the decomposition $E_x = E_+(\xi') + E_-(\xi')$.

Proposition 1.1. *Let x be in M, and ξ' be in $T_x^*M \subseteq T_x^*Y^\sharp$. Then*

$$\sigma(A_+, \xi') = b_-(\xi') , \qquad \sigma(A_-, \xi') = b_+(\xi') .$$

Proof. Since $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi' + z\boldsymbol{n}(x)) = \sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi') + zI$, we may write

$$\sigma(A_+, \xi') = \frac{1}{2\pi i} \int_{\Gamma_+(R_{\xi'})} (\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi') + zI)^{-1}dz .$$

Since $E_+^i(\xi')$ and $E_-^j(\xi')$ are preserved by $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi')$, it is enough to treat them separately. On $E_+^i(\xi')$ (resp. $E_-^j(\xi')$), $\sigma(P, \boldsymbol{n}(x))^{-1}\sigma(P, \xi')$ induces a mapping of the form $\mu_i I - R_i'$ (resp. $\lambda_j I - R_j$) where R_i' and R_j are nilpotent transformations. Since $(\mu_i + z)I - R_i'$ is nonsingular on z with positive imaginary part, $\sigma(A_+, \xi') = 0$ on $E_+(\xi')$. Since R_j is nilpotent,

$$((\lambda_j + z)I - R_j)^{-1} = (\lambda_j + z)^{-1}I + (\lambda_j + z)^{-2}R_j + \cdots + (\lambda_j + z)^{-t}R_j^{t-1}$$

for sufficiently large t. Hence $\sigma(A_+, \xi') = I$ on $E_-(\xi')$ because the orientation on $\Gamma_+(R_{\xi'})$ is anti-clockwise. Hence $\sigma(A_+, \xi') = b_-(\xi')$. Noting that the orientation on $\Gamma_-(R_{\xi'})$ is clockwise, we find similarly that $\sigma(A_-, \xi') = b_+(\xi')$.

In the course of our argument we shall use the following theorem (cf. [2]), which is based on K. Friedrich's proof of P. D. Lax-L. Nirenberg theorem on generalized Garding inequality (cf. also [3, Theorem 1.33]):

Theorem 1.4. *Let $g(x)$ be a real valued C^∞ function on $\boldsymbol{R}^n$ with compact support such that $g(x) \geq 0$, $\int g(x)dx = 1$, and $g(-x) = g(x)$. Let $J(x, \xi)$ be a $m \times m$ matrix of symbols of a pseudo-differential operator of order l. Assume that $J(x, \xi) = 0$ for x outside a compact subset of $\boldsymbol{R}^n$ and that*

$$\langle J(x,\xi)u, u\rangle \geq 0$$

for all $u \in C^m$. *Then for all* $u \in C_0^\infty(R^n, C^m)$,

$$\mathscr{R}\langle J(x,D)u, u\rangle \geq -\mathscr{R}\langle L(x,D)u, u\rangle ,$$

where $L(x,\xi)$ *is a* $m \times m$ *matrix of symbols of pseudo-differential operator of order* $l-1$ *given by*

$$L(x,\xi) = \sum \tfrac{1}{2}(1+|\xi|^2)^{\frac{1}{2}} a_{jk}\partial^2 J(x,\xi)/\partial\xi_i\partial\xi_k + \sum \tfrac{1}{2}(1+|\xi|^1)^{-\frac{1}{2}} b_{jk}\partial^2 J(x,\xi)/\partial x_i\partial x_k + (\text{a term of order } l - 3/2)\,, \tag{11}$$

where

$$a_{ik} = \int x_i x_k g(x)^2 dx\,, \qquad b_{jk} = \int (\partial g/\partial x_i)(\partial g/\partial x_k) dx\,. \tag{12}$$

2. Reduction to the characteristics

Let L be a vector subbundle of E. Our problem is to find conditions on P under which we have the following estimate: For a constant $c > 0$,

$$\|Pu\|_0 + \|u\|_0 \geq c\|u\|_{\frac{1}{2}}$$

for all $u \in C^\infty(\bar{Y}, E)$ with boundary condition $u|M \in C^\infty(M, L)$. In view of Theorem 1.3, the problem is equivalent to finding conditions under which we have the following estimate: For a constant $c > 0$,

$$\|A_+\gamma\|_{\frac{1}{2}} + \|\gamma\|_{-\frac{1}{2}} \geq c\|\gamma\|_0$$

for all $\gamma \in C^\infty(M, L)$. Pick a Riemannian metric on M and denote by R (resp. by $R_{\frac{1}{2}}$) a pseudo-differential operator $C^\infty(M,E) \to C^\infty(M,E)$ such that $\sigma(R,\xi) = |\xi|I$ (resp. $\sigma(R_{\frac{1}{2}}, I) = |\xi|^{\frac{1}{2}}I$), where $|\xi|$ is the length of ξ by the Riemannian metric. We may choose $R_{\frac{1}{2}}$ in such a way that it preserves $C^\infty(M,L)$. Namely, take the orthogonal complement, say K, of L in E and replace $R_{\frac{1}{2}}$ by $\rho_L R_{\frac{1}{2}} \rho_L + \rho_K R_{\frac{1}{2}} \rho_K$, where ρ_L (resp. ρ_K) is the projection to L (resp. to K). Now since the norm $\|\gamma\|_{s+\frac{1}{2}}$ is equivalent to $\|R_{\frac{1}{2}}\gamma\|_s$, by applying the above estimate to $R_{\frac{1}{2}}\gamma$ ($\gamma \in C^\infty(M,L)$) and noting that $R_{\frac{1}{2}}AR_{\frac{1}{2}} - AR$ is an operator of order 0 we see that the estimate is equivalent to the following: For a constant $c > 0$,

$$\|A_+R\gamma\|_0 + \|\gamma\|_0 \geq c\|\gamma\|_{\frac{1}{2}}$$

for all $\gamma \in C^\infty(M,L)$. We will work on this form of the problem. A_+R is a pseudo-differential operator of order 1. More generally, we fix once and for all throughout this section a pseudo-differential operator of order 1

$$A: C^\infty(M, L) \to C^\infty(M, E) ,$$

and ask for conditions under which

$$\|A\gamma\|_0 + \|\gamma\|_0 \geq c\|\gamma\|_{\frac{1}{2}} \tag{1}$$

for all $\gamma \in C^\infty(M, L)$. Assume that A has a leading homogenous symbol, and note that we apply the result to the case where A is the restriction to $C^\infty(M, L)$ of A_+R. By using the partition of unity we see that the estimate (1) is equivalent to the localized ones: For each $x^0 \in M$ there is a neighborhood U of x^0 in M such that the estimate (1) holds for all $\gamma \in C_0^\infty(U, L)$.

If $\sigma(A, \xi)$ is injective for all nonzero $\xi \in T^*M$, then A is elliptic and we have the estimate. Therefore the conditions shall be related to the kernel of the symbol map $\sigma(A): L \otimes T^*M \to E$.

Definition 2.1. $\xi \in T^*M$ is called a characteristic of A if and only if ξ is nonzero and $\sigma(A, \xi)$ is not injective. We denote by $C(A)$ the set of characteristic of A with unit length.

Definition 2.2. We say that characteristics of A are smooth if and only if for each x^0 in M there is an open neighborhood U of x^0 in M satisfying the following conditions: (1) $C(A) \cap \pi^{-1}(U)$, where π is the projection $T^*M \to M$, is a sum of a finite number of disjoint submanifolds $C^1, \cdots, C^s$ of $\pi^{-1}(U) \subseteq T^*M$, (2) for each $\lambda = 1, 2, \cdots, s$, π induces a mapping with constant rank of C^λ onto a submanifold $'C^\lambda$ of U ($'C^1, \cdots, 'C^s$ may not be disjoint), (3) for each λ, dim (ker $\sigma(A, \xi)$) is constant for $\xi \in C^\lambda$.

If this is the case, for each x in $'C^\lambda$, $S^*M \cap \pi^{-1}(x) \cap C^\lambda = C_x^\lambda$ forms a submanifold of S^*M, where S^*M denotes the bundle of cotangent vectors of unit length. For $\zeta \in C_x^\lambda$, we set $W^\lambda(\zeta) = \ker \sigma(A, \zeta) \subseteq L_x$. Clearly

$$W^\lambda(\zeta) = \ker \sigma(A, \zeta)^*\sigma(A, \zeta) . \tag{2}$$

Since $\sigma(A, \zeta)^*\sigma(A, \zeta)$ is hermitian and positive semi-definite, we may assume that all of its nonzero eigenvalues are larger than $\delta > 0$. Hence there is an open neighborhood $\mathscr{U}^\lambda$ of $C^\lambda = C^\lambda \cap S^*M$ in S^*M such that, for any $\xi \in \mathscr{U}^\lambda$, the sum $W^\lambda(\xi)$ of the eigen-subspaces of $\sigma(A, \xi)^*\sigma(A, \xi)$ with eigen-values less than $\frac{1}{2}\delta$ depends differentiably on ξ. By construction this new $W^\lambda(\xi)$ for $\xi \in C^\lambda$ coincides with the old $W^\lambda(\zeta)$ when $\zeta = \xi$. We may assume without loss of generality

$$\mathscr{U}^\lambda \cap \mathscr{U}^\mu = \emptyset \qquad (\lambda \neq \mu) . \tag{3}$$

Denote by $\rho_1^\lambda(\xi)$ ($\xi \in \mathscr{U}^\lambda$) the projection of L_x into $W^\lambda(\xi)$, and by $\rho_2^\lambda(\xi)$ the projection of L_x into the orthogonal complement of $W^\lambda(\xi)$ in L_x. Thus

$$\rho_1^\lambda(\xi) + \rho_2^\lambda(\xi) = I_x \qquad (\xi \in \mathscr{U}^\lambda \cap T_x^*M) , \tag{4}$$

where I_x is the identity map of L_x. Choose an open neighborhood $'\mathscr{V}$ of $C^1 \cup \cdots \cup C^s$ in S^*U such that

$$(5) \qquad \text{the closure of } '\mathscr{V} \subseteq \mathscr{U}^1 \cup \cdots \cup \mathscr{U}^s ,$$

and denote by $\mathscr{V}$ the complement of the closure of $'\mathscr{V}$ in S^*U. Thus $\{\mathscr{U}^1, \cdots, \mathscr{U}^s, \mathscr{V}\}$ is a covering of S^*U. Let $\{\varphi^1, \cdots, \varphi^s, \varphi\}$ be a C^∞ partition of unity for S^*U with respect to the covering. Thus

$$(6) \qquad 1 = \varphi + \varphi^1 + \cdots + \varphi^s , \quad \operatorname{Supp} \varphi^\lambda \subseteq \mathscr{U}^\lambda , \quad \operatorname{Supp} \varphi \subseteq \mathscr{V} .$$

We extend φ^λ to a C^∞ function on T^*U such that $\varphi^\lambda(\xi) = \varphi^\lambda(\xi/|\xi|)$ for $|\xi| > \frac{1}{2}$ and $=0$ on a neighborhood of origin, and similarly for φ, $\rho_1^\lambda(x, \xi)$, and $\rho_2^\lambda(x, \xi)$. (We call this process an extension by homogeneity of degree 0 with cut-off near origin.)

We fix a chart $x = (x_1, \cdots, x_{n-1})$ on U such that the origin represents x^0. Until further notice, $\xi = (\xi_1, \cdots, \xi_{n-1})$ will denote elements in $\boldsymbol{R}^{n-1}$. The cotangent vector $\xi_1 dx_1 + \cdots + \xi_{n-1} dx_{n-1}$ at x will be denoted by (x, ξ). We fix a local trivialization of L and E over $U, L|U \approx U \times L_0, E|U \approx U \times E_0$, and regard $\sigma(A, (x, \xi))$ as a Hom (L_0, E_0)-valued function. We take a C^∞ function $f(\xi)$ which is 1 for $|\xi| > \frac{1}{2}$ and zero near origin, and set

$$(7) \qquad a(x, \xi) = f(\xi)\sigma(A, (x, \xi))\,|\,L_0 .$$

The pseudo-differential operator with symbol $q(x, \xi)$ will be denoted by $q(x, D)$. Thus our problem is to find conditions under which

$$Q(u) = \|a(x, D)u\|^2 + \|u\|^2 \geq c(\|u\|_{\frac{1}{2}})^2 \qquad (u \in C_0^\infty(U^0, U^0 \times L_0))$$

for a sufficiently small neighborhood U^0 of x^0 in U.

We will make the following conventions in the remainder of this section: If $r(x, D)$ is a pseudo-differential operator, then

$$\{r(x, D)q(x, D)\}$$

denotes the pseudo-differential operator with symbol $r(x, \xi)q(x, \xi)$, while $r(x, D)q(x, D)$ will denote the composition of the two. $\{\ \}$ will be used in the same way not only for two but for any finite number of pseudo-differential operators, as well as for matrix of pseudo-differential operators. c will denote a positive constant, and T_l a pseudo-differential operator of order l, both of which may be different from formula to formula. It will be used when the exact form of its symbol is not important.

We fix an open neighborhood U' of x_0 in U such that the closure of U' is in U. This is due to the following technical reason: Let $\theta(x)$ be a C^∞ function which is 1 on U' and with compact support in U. Let S be a pseudo-differential operator and u be in $C_0^\infty(U', U' \times \boldsymbol{C})$. Then $Su = \theta Su + (1 - \theta)Su$ and $(1 - \theta)S$ is an operator of order $-\infty$ on $C_0^\infty(U', U' \times \boldsymbol{C})$ because of the quasi-locality of pseudo-differential operators. Thus, up to an error of an operator of order $-\infty$, we may replace S by θS which has a compact support. Therefore, as long as we consider $u \in C_0^\infty(U', U' \times L_0)$, we may use formulas for symbols of compositions and adjoints of pseudo-differential operators with compact support even if we are handling operators which may not have compact support, provided we allow errors of order $-\infty$. In our estimate problem, such errors can be always absorbed in $Q(u)$. Since the argument is obvious, we will omit this part of the argument.

Lemma 2.1. *Let $\psi(x, \xi)$ (resp. $\psi_1(x, \xi)$) be* Hom (L_0, L_0) *(resp.* Hom (E_0, E_0)*) valued symbol of pseudo-differential operator of order* 0, *and assume $a(x, \xi)\psi(x, \xi) = \psi_1(x, \xi)a(x, \xi)$ outside a bounded subset of $U \times \boldsymbol{R}^{n-1}$. Then*

$$Q(u) \geq c\|a(x, D)\psi(x, D)u\|^2 \qquad (u \in C_0^\infty(U', U' \times L_0)) .$$

Proof. By assumption, $a(x, D)\psi(x, D)u = \psi_1(x, D)a(x, D)u + T_0 u$ for $u \in C_0^\infty(U', U' \times L_0))$. Hence our lemma follows from the continuity of $\psi_1(x, D)$.

Lemma 2.2. *Let $\psi(x, \xi)$ and $\psi_1(x, \xi)$ be as in Lemma* 2.1, *and assume further that there is a* Hom (L_0, V) *valued symbol $\theta(x, \xi)$ of pseudo-differential operator of order* 1, *where V is a vector space, such that $a(x, \xi) \oplus \theta(x, \xi)$ is a* Hom $(L_0, L_0 \oplus V)$ *valued elliptic symbol, and such that $\theta(x, \xi)\psi(x, \xi) = 0$ outside a bounded subset of $U \times \boldsymbol{R}^{n-1}$. Then*

$$Q(u) \geq c(\|\psi(x, D)u\|_1)^2 \qquad (u \in C_0^\infty(U', U' \times L_0)) .$$

Proof. By our assumption, $\theta(x, D)\psi(x, D)$ is an operator of order $-\infty$, so that $\|\psi(x, D)u\|_1 \leq c\|(a(x, D) \oplus \theta(x, D))\psi(x, D)u\| + c\|u\| \leq c\|a(x, D)\psi(x, D)u\| + c\|u\|$. Hence by Lemma 2.1 we obtain our inequality.

Lemma 2.3. *Let $f(x, \xi)$ be a symbol of pseudo-differential operator of order* 0, *and assume that* Supp $f \cap \pi^{-1}\pi(C(A))$ *is bounded, where π is the projection: $T^*M \to M$. Then*

$$Q(u) \geq c(\|f(x, D)u\|_1)^2 \qquad (u \in C_0^\infty(U', U' \times L_0)) .$$

Proof. We set $\psi(x, \xi) = f(x, \xi)I_{L_0}$, $\psi_1(x, \xi) = f(x, \xi)I_{E_0}$, where I with an index is the identity map of the index. Pick a C^∞ function $g(x, \xi)$, homogenous of degree 0 for $|\xi| > \frac{1}{2}$, such that

$$C(A) \cap \pi^{-1}(U) \subseteq \text{Supp}\, g\ , \qquad \text{Supp}\, g \cap \text{Supp}\, f \text{ is bounded.}$$

Set $\theta(x, \xi) = g(x, \xi)|\xi|I_{L_0}$. Then assumptions in Lemma 2.2 with $V = L_0$ are satisfied, and hence by Lemma 2.2 we obtain our inequality.

Lemma 2.5. *There are* Hom (E_0, E_0) *valued C^∞ functions $'\rho_1^\lambda(x, \xi)$, $'\rho_2^\lambda(x, \xi)$ of homogenous of order* 0 *for $|\xi| > \frac{1}{2}$ such that for all $(x, \xi) \in \mathscr{U}^\lambda$ outside a bounded subset of $\pi^{-1}\pi\mathscr{U}^\lambda$,*

$$'\rho_1^\lambda(x, \xi)a(x, \xi) = a(x, \xi)\rho_1^\lambda(x, \xi) ,$$
$$'\rho_2^\lambda(x, \xi)a(x, \xi) = a(x, \xi)\rho_2^\lambda(x, \xi) .$$

Proof. We consider only $(x, \xi) \in \mathscr{U}^\lambda$ with $|\xi| > \frac{1}{2}$. $\rho_2^\lambda(x, \xi)$ is the projection into the orthogonal complement of $W^\lambda(x, \xi)$ which depends differentiably on (x, ξ) and on which $a(x, \xi)$ is injective. Hence the projection $'\rho_2^\lambda(x, \xi)$ of E_0 into the image of $a(x, \xi)\rho_2^\lambda(x, \xi)$ depends differentiably on (x, ξ), and we have

$$(8) \qquad '\rho_2^\lambda(x, \xi)a(x, \xi)\rho_2^\lambda(x, \xi) = a(x, \xi)\rho_2^\lambda(x, \xi) .$$

Since $\rho_1^\lambda(x, \xi)$ and $\rho_2^\lambda(x, \xi)$ are projections into the sums of different eigen-subspaces of $a(x, \xi)^*a(x, \xi)$, the images of $a(x, \xi)\rho_1^\lambda(x, \xi)$ and $a(x, \xi)\rho_2^\lambda(x, \xi)$ are orthogonal. Therefore $'\rho_2^\lambda(x, \xi)a(x, \xi)\rho_1^\lambda(x, \xi) = 0$ so that (4) and (8) imply that $'\rho_2^\lambda(x, \xi)a(x, \xi) = a(x, \xi)\rho_2^\lambda(x, \xi)$. Since $a(x, \xi)\rho_1^\lambda(x, \xi) = a(x, \xi) - a(x, \xi)\rho_2^\lambda(x, \xi) = a(x, \xi) - '\rho_2^\lambda(x, \xi)a(x, \xi)$, $'\rho_1^\lambda(x, \xi) = I_x - '\rho_2^\lambda(x, \xi)$ satisfies the first equality.

Lemma 2.6. *Let $f^\lambda(x, \xi)$ be a symbol of pseudo-differential operator of order* 0 *such that* supp $f^\lambda \subseteq \pi^{-1}\pi\mathscr{U}^\lambda$. *Then*

$$Q(u) \geq c\|\{f^\lambda(x, D)\rho_2^\lambda(x, D)\}u\|_1 .$$

Proof. We set $\psi(x, \xi) = f^\lambda(x, \xi)\rho_2^\lambda(x, \xi)$, $\psi_1(x, \xi) = f^\lambda(x, \xi)'\rho_2^\lambda(x, \xi)$, and $\theta(x, \xi) = |\xi|(\varphi^1(x, \xi)\rho_1^1(x, \xi) \oplus \cdots \oplus \varphi^s(x, \xi)\rho_1^s(x, \xi))$. Then they satisfy the conditions in Lemma 2.2 where V is the direct sum of s-copies of E_0. Hence we have our lemma.

Choose a submanifold N^λ of U transversal to $'C^\lambda$ and intersecting $'C^\lambda$ only at x^0. For w in N^λ, $w = (w_1, \cdots, w_{n-1})$ will denote the coordinate of w with respect to the chart $x = (x_1, \cdots, x_{n-1})$, and similarly for y in $'C^\lambda$. By shrinking N^λ and $'C^\lambda$ if necessary, we may assume that

$$N^\lambda \times {}'C^\lambda \ni (w, y) \to w + y = (w_1 + y_1, \cdots, w_{n-1} + y_{n-1}) \in U^\lambda$$

is a diffeomorphism onto an open neighborhood U^λ of x^0. By means of the projection in terms of the above direct product structure of U^λ, we often regard $w \in N^\lambda$ and $y \in {}'C^\lambda$ as functions $w(x)$ and $y(x)$ of $x \in U^\lambda$. Thus

$$(9) \qquad x = w(x) + y(x) .$$

To proceed further we assume that $\pi\colon C^\lambda \to {}'C^\lambda$ is bijective. Thus over each $y \in {}'C^\lambda$ there is a unique characteristic $\zeta^\lambda(y)$ in C^λ. Denote by $\chi(x, \xi)$ the projection of ξ into the orthogonal complement of the subspace generated by $\zeta^\lambda(y(x))$. Thus

$$(10) \qquad \xi = \langle\zeta^\lambda(y(x)), \xi\rangle\zeta^\lambda(y(x)) + \chi(x, \xi) .$$

By definition $\chi(x, \xi) = \chi(y(x), \xi)$.

Lemmas 2.3 and 2.6 mean that we have enough information for $\varphi(x, D)u$ and $\{\varphi^\lambda(x, D)\rho_2^\lambda(x, D)\}u$. It remains to study $\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u$. We start by analyzing $\varphi^\lambda(x, \xi)a(x, \xi)\rho_1^\lambda(x, \xi)$, where ξ whose direction is close to that of

$\zeta^\lambda(y(x))$ matters. In fact, $|\xi|^{-1}\chi(x, \xi)$ on Supp φ^λ can be made as small as we wish by choosing φ^λ such that its support is in a very small neighborhood of C^λ. So we expand $a(x, \xi)\rho_1^\lambda(x, \xi)$ as Taylor series in (w, χ) at $(y, \langle\zeta^\lambda(y), \xi\rangle\zeta^\lambda(y))$. For the simplicity of notations we suppress x in $y(x)$ and $w(x)$. Thus we have

$$\begin{aligned} a(x, \xi)\rho_1^\lambda(x, \xi) &= \langle\zeta^\lambda(y), \xi\rangle a(y + w, \zeta^\lambda(y) + \langle\zeta^\lambda(y), \xi\rangle^{-1} \\ &\quad \cdot\chi(y, \xi))\rho_1^\lambda(y + w, \zeta^\lambda(y) + \langle\zeta^\lambda(y), \xi\rangle^{-1}\chi(y, \xi)) \\ &= \sum_{j=1}^{n-1} f^j(y)w_j\langle\zeta^\lambda(y), \xi\rangle + \sum_{j=1}^{n-1} g^j(y)\chi_j(y, \xi) \\ &\quad + H_1(x, \xi) , \end{aligned} \tag{11}$$

where

$$\begin{aligned} f^j(y) &= a^{(j)}(y, \zeta^\lambda(y))\rho_1^\lambda(y, \zeta^\lambda(y)) + a(y, \zeta^\lambda(y))\rho_1^{\lambda(j)}(y, \zeta^\lambda(y)) , \\ g^j(y) &= a^j(y, \zeta^\lambda(y))\rho_1^\lambda(y, \zeta^\lambda(y)) + a(y, \zeta^\lambda(y))\rho_1^{\lambda j}(y, \zeta^\lambda(y)) , \end{aligned} \tag{12}$$

$$\begin{aligned} a^{(j)}(x, \xi) &= \partial a(x, \xi)/\partial x_j , & a^j(x, \xi) &= \partial a(x, \xi)/\partial \xi_j , \\ \rho_1^{\lambda(j)}(x, \xi) &= \partial \rho_1^\lambda(x, \xi)/\partial x_j , & \rho_1^{\lambda j}(x, \xi) &= \partial \rho_1^\lambda(x, \xi)/\partial \xi_j . \end{aligned} \tag{13}$$

$H_1(x, \xi)$ is of class C^∞, order 1 for $|\xi| > \frac{1}{2}$, and at least order 2 in $w_1, \cdots, w_{n-1}$, $\chi_1, \cdots, \chi_{n-1}$, that is, we can write in a neighborhood of C^λ

$$\begin{aligned} H_1(x, \xi) = \sum_{j,k=1}^{n-1} (H_1^{jk}(x, \xi)w_jw_k + H_0^{jk}(x, \xi)w_j\chi_k(x, \xi) \\ + H_{-1}^{jk}(x, \xi)\chi_j(x, \xi)\chi_k(x, \xi)) . \end{aligned} \tag{14}$$

We regard $f^1(y) = \cdots = f^{n-1}(y) = 0$ when ${}'C^\lambda = U$. We may assume thatt he expression is valid on Supp φ^λ, and set

$$F^\lambda(y; w, \chi) = \sum_{j=1}^{n-1} (f^j(y)w_j + g^j(y)\chi_j) \tag{15}$$

so that by (11)

$$\begin{aligned} a(x, \xi)\rho_1^\lambda(x, \xi) = F^\lambda(y; w\langle\zeta^\lambda(y), \xi\rangle, \chi(y, \xi))\rho_1^\lambda(x, \xi) \\ + H_1(x, \xi)\rho_1^\lambda(x, \xi) . \end{aligned} \tag{16}$$

Since $a(y, \zeta^\lambda(y))\rho_1^\lambda(y, \zeta^\lambda(y)) = 0$ and $\chi(y, \zeta^\lambda(y)) = 0$,

$$F^\lambda(y; 0, \zeta^\lambda(y))\rho_1^\lambda(y, \zeta^\lambda(y)) = 0 . \tag{17}$$

Set

$$F^\lambda = F^\lambda(y(x), w\langle\zeta^\lambda(y), D\rangle, \chi(y, D)) \tag{18}$$

which is a differential operator. By (16) we have

$$\begin{aligned} &a(x, D)\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u \\ &\quad = F^\lambda\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u + H_1(x, D)\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\} + T_0u . \end{aligned} \tag{19}$$

Hence for each $0 < \delta < 1$

$$\|a(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 \\ = (1-\delta)\|F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + \mathscr{R}\langle K_\delta^\lambda(x,D)u, u\rangle , \tag{20}$$

where

$$\begin{aligned} K_\delta^\lambda(x,D) = {} & \delta\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}^* F^{\lambda *} F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} \\ & + (H_1(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} + T_0)^* \\ & \qquad (H_1(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} + T_0) \\ & + \{\varphi^\lambda(x,D)\rho_1^\lambda(X,D)\}^* F^{\lambda *}(H_1(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} + T_0) \\ & + (H_1(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} + T_0)^* F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\} . \end{aligned} \tag{21}$$

Denote the homogenous order 2 part of $K_\delta^\lambda(x,\xi)$ by $K_{\delta 2}^\lambda(x,\xi)$. Then

$$K_{\delta 2}^\lambda(x,\xi) = \varphi^\lambda(x,\xi)^2\rho_1^\lambda(x,\xi)\delta F^\lambda(y; w,\chi)^* F^\lambda(y; w,\chi) \\ + H_2^\lambda(x,\xi))\rho_1^\lambda(x,\xi) , \tag{22}$$

where

$$H_2^\lambda(x,\xi) = H_1(x,\xi)^* H_1(x,\xi) + F^\lambda(y; w\langle\zeta^\lambda(y),\xi\rangle,\chi)^* H_1(x,\xi) \\ + H_1(x,\xi)^* F^\lambda(y; w\langle\zeta^\lambda(y),\xi\rangle,\chi) , \tag{23}$$

which is of at least order 3 in $w_1, \cdots, w_{n-1}, \chi_1, \cdots, \chi_{n-1}$. Let $K_{\delta 1}^\lambda(x,\xi)$ be the homogenous part of order 1 in $K_\delta^\lambda(x,\xi)$. To find it, we apply to (20) the formula for symbols of compositions and adjoint of pseudo-differential operators. By Lemma 2.3 the terms whose support does not touch $C^1 \cup \cdots \cup C^s$ can be absorbed in $Q(u)$, so that their exact form is not important to us. For example, derivatives of $\varphi^\lambda(x,\xi)$ is of this type. Also exact form of terms containing δ and terms containing w_j or χ_j turns out to be not important to us. By examining terms we find that they are always one of the above types. Thus

$$K_{\delta 1}^\lambda = \varphi^\lambda(x,\xi)^2(\delta T_1^\lambda(x,\xi) + L_1^\lambda(x,\xi)) + S_1^\lambda(x,\xi) , \tag{24}$$

where

$$L_1^\lambda(x,\xi) = \textstyle\sum_{j=1}^{n-1} L_{1j}(x,\xi)w_j + \sum_{j=1}^{n-1} L_{0j}(x,\xi)\chi_j(y,\xi) ,$$

$$\operatorname{Supp} S_1^\lambda \cap (C^1 \cup \cdots \cup C^s) = \emptyset . \tag{25}$$

We note that $T_1^\lambda(x,\xi)$ and $L_1^\lambda(x,\xi)$ are independent of the choice of $\varphi^\lambda(x,\xi)$, but $S_1^\lambda(x,\xi)$ is not.

Lemma 2.7. *Let $\varepsilon > 0$. Then there is δ_ε such that for any $\delta_\varepsilon > \delta > 0$ we can find a constant $C_{\varepsilon,\delta}$ (which also depends on a choice of φ^λ) such that*

$$C_{\varepsilon,\delta}Q(u) + \varepsilon(\|u\|_{\frac{1}{2}})^2 \geq (1-\delta)\|F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + \mathscr{R}\langle K_{\delta 2}^\lambda(x,D)u, u\rangle$$

for $u \in C_0^\infty(U, U \times L_0)$, *provided that* U *is sufficiently small and that the support of* φ^λ *is sufficiently close to* C^λ.

Proof. Let $\varepsilon_1 > 0$. We may choose U and φ^λ such that $|w_j|$ on U and $|\chi_j(y,\xi)|/|\xi|$ on Supp φ^λ are sufficiently small so that $\|\varphi^\lambda(x,D)^2 L_1^\lambda(x,D)u\|_{-\frac{1}{2}} \leq \varepsilon_1\|u\|_{\frac{1}{2}}$ for $u \in C_0^\infty(U, U \times L_0)$. Then

$$\begin{aligned}&|\langle K_{\delta 1}^\lambda(x,D)u, u\rangle| \\ &\quad \leq \delta|\langle\{\varphi^\lambda(x,D)T_1^\lambda(x,D)\}u, u\rangle| + \varepsilon_1\|u\|_{\frac{1}{2}} + |\langle S_1^\lambda(x,D)u, u\rangle| \\ &\quad \leq (\delta C + \varepsilon_1)\|u\|_{\frac{1}{2}} + |\langle S_1^\lambda(x,D)u, u\rangle| \leq (\delta C + \varepsilon_1)\|u\|_{\frac{1}{2}} + CQ(u)\end{aligned}$$

by Lemma 2.3. Therefore by (21)

$$\begin{aligned}&\|a(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + CQ(u) + (\delta c + \varepsilon_1)\|u\|_{\frac{1}{2}} \\ &\quad \geq (1-\delta)\|F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + \mathscr{R}\langle K_{\delta 2}^\lambda(x,D)u, u\rangle .\end{aligned}$$

On the other hand Lemmas 2.1 and 2.4 imply that

$$\|a(x,D)\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\| \leq C_1 Q(u) .$$

Thus

$$\begin{aligned}&(C + C_1)Q(u) + (\delta c + \varepsilon_1)\|u\|_{\frac{1}{2}} \\ &\quad \geq (1-\delta)\|F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + \mathscr{R}\langle K_{\delta 2}^\lambda(x,D)u, u\rangle .\end{aligned}$$

Hence it is enough to choose δ_1 so that $\delta_1 c < \varepsilon$, and set $\varepsilon_1 = \varepsilon - \delta c$. q.e.d.

In view of Lemma 2.7, it may be reasonable to seek conditions under which

$$\langle K_{\delta 2}^\lambda(x,D)u, u\rangle + C_\varepsilon Q(u) \geq -\varepsilon\|u\|_{\frac{1}{2}}^2 ,$$

$$\|F^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + Q(u) \geq c\|\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|_{\frac{1}{2}}^2 .$$

In the remainder of this section we formulate conditions for the first of the inequality, and those for the second will be discussed in § 3.

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